Rethinking the teaching of systems of equations

Highlighting multiple representations of systems of equations

In a recent issue of Mathematics Teacher, Glenda Lappan (2006) reminded us of the role this journal has played in the mathematics education community by raising issues of concern to mathematics educators, especially by addressing what we teach in mathematics and why. Our article focuses on the nature of the mathematics taught in the classroom and explores the conceptual richness of systems of equations.

Our work in classrooms and with teachers in the context of professional development reveals that the solution of systems of equations is seen as a relatively straightforward topic in the high school curriculum, a continuation of solving equations and problems in one variable. From a review of textbooks, Web-based lessons, and other curricular documents, we found that solving systems of equations is often introduced through the use of graphs, and then the focus is turned to completing exercises and solving problems by using various algebraic methods. Even though graphs are the means of introducing the concept of systems of equations and their solutions, graphical approaches are quickly set aside for the simplicity and strength of algebraic methods. As students grow accustomed to following algebraic procedures to manipulate systems symbolically, they seldom go back to graphical (or other) approaches to find different ways of thinking about or solving the systems. Ganon and Shultz (2006) made a similar observation: “Although the geometry motivates the existence of a solution, the student quickly reverts to an ‘algebra’ mode of thinking” (p. 189). Graphing, therefore, becomes a secondary or optional approach rather than one at the core of the study of systems of equations.
What are the implications for student understanding when, during the teaching of systems of equations, algebraic manipulation is emphasized over graphical or other conceptual approaches? Sfard and Linchevski (1994) discuss some repercussions of focusing on algebraic manipulations. They illustrate how students dependent on algebraic techniques are sometimes unable to make sense of the solutions they come up with and become lost when their mechanical strategy fails them. For example, they gave high school students the following system of equations: $2(x - 3) = 1 - y$ and $2x + y = 7$. Some students arrived at the same equation for both by using algebraic manipulations ($2x + y = 7$) and then substituted one equation into the other to obtain $7 = 7$. These students concluded that $x$ was equal to zero and resubstituted the $x$ in one equation to obtain $y = 7$, thereby giving only the coordinate $(0, 7)$ as the solution. These students went through a series of mechanical manipulations to arrive at an “answer,” but they did not reflect on the “solution” $7 = 7$ or question its meaning, nor did they look back at the problem to check whether their solution satisfied it.

As NCTM’s Principles and Standards for School Mathematics (2000) articulates, mathematics is much more than solving problems by using mechanized procedures and following prescriptions for algorithms. It involves not only well-rehearsed procedures but also continuous reasoning, sense making, reflection, and critical evaluation. However, from our analysis of textbooks and curricular materials, the study of systems of equations...
often appears to focus on algebraic manipulations, problems that involve linear equations with a single point of intersection, and alternative algebraic procedures for solving systems. Systems of equations is a much richer topic than that: It is a powerful mathematical tool to model, represent, and solve problems. In this article, we ask how the teaching of systems of equations can be enhanced to allow students to study this topic from a more comprehensive perspective. To that end, we have produced a concept map similar to those discussed extensively by Skemp (1987). Our concept map for systems of equations illustrates the potential multi-dimensionality of the topic (see fig. 1).

**A CONCEPT MAP FOR SYSTEMS OF EQUATIONS**

The concept map reflects four possible areas of focus for teaching and learning systems of equations: (1) the meaning of a system of equations; (2) ways of representing a system of equations; (3) ways of solving a system of equations; and (4) interpreting solutions of a system of equations. We will elaborate on each.

**The Meaning of a System of Equations**

A system can be thought of as a set of objects working together. To work properly, a system must simultaneously satisfy the required conditions or constraints imposed by the elements that constitute it and the context in which it exists. One might say that a system of equations is made up of two or more equations that represent conditions that must be satisfied simultaneously for the system to be operative. A system of equations is not a collection of independent equations but a set of equations that are in relationship.

The decision to model a given situation with a system of equations is made on the basis of its potential strength as a mathematical tool to solve specific problems. Simple algebraic problems (see fig. 2) that can be solved with only one equation should not be included in a unit on systems of equations, because such problems do not show the relevance of using systems of equations. Problems posed for instructional purposes need to demonstrate that using systems of equations has mathematical relevance and enables the solving and modeling of problems that other mathematical methods cannot accommodate or can accommodate only with difficulty. For example, the problem shown in figure 3, by not providing any direct relationship between the number of adults and of children, makes representation with only one variable more difficult and therefore motivates to some extent the progression and transition to using two variables to represent the problem and find the requested amounts.
Cooney and Wiegel (2003) argue that mathematics is seen primarily as a rigid field of study where often there is only one way to solve a particular problem. Against such a view, they argue for a "pluralistic perspective," in which problems are approached from different avenues. Students may encounter systems of equations in many ways. They may be given the algebraic form outright (see fig. 4), a word problem describing a situation (see fig. 5), or a graph (see fig. 6). (Note that in the latter two cases the students themselves must generate the equations.) A system of equations may also be offered within the context of geometry (see fig. 7). Further, a system of equations can be represented with a table of values. Table 1 gives the values of two equations \( y_1 = 54 - 3x \) and \( y_2 = -x + 30 \) for a particular \( x \) (these are the equations for the ticket problem presented in fig. 3). By paying attention to the variation of the \( y \)-values for each equation, students can observe when the two equations are equivalent and analyze the numerical values (see also Taylor [2000] for using lists). This representation can be offered without the algebraic or graphical forms, and teachers often use it as a bridge between the algebraic and the graphical. Although all forms of representation the student may encounter are relevant to the study of mathematics, our research suggests that students are not encountering them all and that, of those they do encounter, the algebraic presentation of systems of equations heavily predominates.

Janvier (e.g., 1987) has argued that all these representations should be presented in the study and teaching of mathematics and that students should be oriented toward and offered plenty of opportunities for moving and translating between them. Continuously addressing these different representations can make the study and understanding of systems of equations more flexible and enable students, in their critical thinking, to move back and forth between representations, making choices informed by relevant criteria rather than simply based on one's familiarity with a particular form.

**Ways of Solving a System of Equations**

One goal of teaching systems of equations in high school mathematics is to provide students with a mathematical tool for solving a particular class of problems. Because of the dominance of algebra in school mathematics, most problem-solving methods are algebraic ones (algebraic methods are seen to be more rigorous than other ones). But the desire for rigor can lead to incomplete and inadequate under-
understanding of the concept of a system of equations, as Sfard and Linchevski's research (1994) illustrates. Hence, in addition to algebraic methods (of substitution, elimination, and comparison), tables of values and graphs are also important means for understanding and solving systems. They expand the student's conception of a system of equations.

Students must be able to explore the variety of methods for solving systems in order to understand each method's strengths and weaknesses; then, when the students work with systems of equations, they can make critical decisions in their mathematical modeling. Moreover, each method requires the use of difficult and rich mathematics and mathematical skills. For example, being able to interpret variations in the graph visually is mathematically significant for understanding the situation graphically, a unique perspective that cannot be assessed by algebraic methods. However, precise numerical values are often out of reach when one is using graphing methods.

Alternatively, using a table of values enables a particular kind of attention to the ordered pairs that other methods do not afford. (As the example in Table 1 illustrates, one can follow the evolution of each equation to draw out information about each equation in the system.) In this sense, an integration of the various methods for modeling and solving problems appears important for a broad and deep understanding of systems of equations. This argument embodies more than the common notion of offering multiple strategies so that students can choose a method for solving a system. As Janvier (e.g., 1987) argued, a robust understanding of the topic requires understanding the transition from one method to another and understanding the interplay between each representation. For this reason, we should be even more explicit about the use of multiple representations in our teaching.

### Interpreting Solutions of a System of Equations

Beyond the strategies and representations emphasized, the meaning attributed to solutions needs to be explored. In other words, how are the solutions interpreted? At the core of any interpretation of a solution is an evaluative and critical orientation to the solution. Curricular materials commonly present systems as having zero, one, or an infinite number of solutions, which represent, respectively, lines having no point of intersection, lines having a single point of intersection, or lines being coincident. There is, however, much more to understanding a solution to a system of equations, and we briefly consider some of these solution "cases."

#### No solution. When graphs of the lines representing the equations do not intersect, there is no "point" of intersection. But what does this statement mean? Does it mean that there is no coordinate that satisfies both equations simultaneously? The fact that there are no points of intersection is the solution to the problem; it describes the relationship between the equations in the system. Graphically, the learner can "see" that the lines are parallel and will not touch each other (see Fig. 8). Algebraically, however, this is not so obvious. Unpacking the solution in terms of the algebraic expression for the equations is useful.

A first indication that there is no solution arises when one observes that the slopes of the two equations are the same but the $y$-intercepts are different. Often, the common slope is not so obvious; it may be “hidden” in the equation. Hence, students need to use algebraic methods to seek any potential points

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of intersection. In doing so, the student arrives at a contradiction (e.g., $3 = 5, 0 = 2$). What does arriving at a contradiction mean? This contradiction surfaces after the student has tried to find a point of intersection. Using a substitution or comparison method, the student arrives at a contradiction after having assumed that there was one point of intersection and having attempted to solve for it. For example, with the equations (1) $2y + 4x = 7$ and (2) $y = -2x - 9$, algebraic methods lead to the contradictions shown in figure 9. Both methods assume that $y$ is the same for both equations and that it can be replaced in both, leading to a contradiction.

Another possibility for arriving at no solution involves studying the context of the problem and its constraints (expressed as the domain and range of the equations). Some contexts are within the domain of the natural numbers, and thus only natural numbers are possible solutions (e.g., only solutions in the first quadrant). If the point of intersection lies outside the first quadrant, then the coordinates of the point are not an adequate solution to the problem. The problem shown in figure 10 also illustrates and contextualizes this situation; here, an intersecting point lies outside the first quadrant, where only positive numbers are acceptable (see fig. 10). The difficulty students might experience, both graphically and algebraically, is that they might arrive at a unique solution, which, however, is contextually impossible.

A unique solution. A second case among potential solutions to systems of equations is the unique solution. This is obtained when the lines of the system intersect only once, that is, when a single point satisfies both equations simultaneously. From our review of textbooks and other materials, the unique solution represents the situation students encounter most frequently and thus may be a significant cause of their difficulties with algebraic manipulations and interpretations of solutions: Students come to expect a single unique point of intersection as the solution and consequently do not question a contradiction or other apparent anomalies (Sfard and Linchevski 1994). This is not to say that students should not encounter situations involving a unique solution; however, as teachers we must make choices for tasks and exercises that demand that students reflect on the nature and appropriateness of the solution.

Infinite number of solutions. The third case is the infinite number of solutions, when lines are coincident. The subtlety that students often overlook in this situation is that although there are an infinite number of intersection points, not every point on the plane will satisfy the system. A more appropriate response might be that an infinite number of points lie along $y = mx + b$ (see Sfard and Linchevsky 1994). The textbooks we reviewed recognized that an infinite number of points of solutions lie along the coincident lines. However, the answers provided at the back of the textbook often present a simple formulation such as “infinity of solutions.” Constraints can also play a role in modifying the case of an infinite number of solutions. For example, if only positive values are appropriate to the context, then the answer could potentially be restricted to a particular range of values. In this first case, although the $x$ variable is restricted to values greater than zero, many solutions are possible (see fig. 11). The problem shown in figure 12 illustrates a type of problem having both a restricted domain and a range that still has an infinite solution. Alternatively, the solution could also be infinite between two values—for example, between 0 and 8 (see fig. 13) or between 0 and 35 (see fig. 14). Further, if the domain is counting numbers, then again, in spite of coincident lines, we would have a finite number of answers: $(0, y_1), (1, y_2), (2, y_3), \ldots, (7, y_8), (8, 0)$. The contextualized

\[ \begin{align*}
\text{Substitution Method} & \quad \text{Comparison Method} \\
2(-2x - 9) + 4x = 7 & \quad (1) y = -2x + 7/2 \\
-4x - 18 + 4x = 7 & \quad (2) y = -2x - 9 \\
-18 = 7 & \quad -2x + 7/2 = -2x - 9 \\
\end{align*} \]

Fig. 9 Contradictions obtained when using algebraic methods for systems of equations with no point of intersection

We offer two ways to a salesman to calculate his salary. In the first, he receives a base salary of $50 and $2 for each book he sells. In the second, he receives a base salary of $40 and $1.75 per book sold. For how many books sold will the salesman have the same salary independent of how his salary is calculated?

Fig. 10 A contextualized problem with an intersection point lying outside the possible range
I am thinking of two positive numbers. The first one is 10 more than a third of the other one. Also, if I triple the first one and subtract the second number from it, I obtain 30. What are the two numbers?

**Fig. 12** A contextualized problem having an infinite solution for which only positive answers are acceptable

The problem shown in **figure 15** illustrates a similar situation.

Textbooks and curricular materials do offer all three cases (zero, one, or an infinite number of solutions). But the possible variations within each solution type are seldom explored, and each type is often not equally represented.

**Continuous and discrete situations.** The issue of continuous and discrete situations is an ever-present concern in mathematical modeling. Most situations encountered in school mathematics are continuous (Even 1993). Therefore, students come to expect that all situations are continuous and represent them (discrete or not) with continuous graphs. Contino (1995) notes this tendency among students studying functions: “As is often done in mathematics classes, [students] ignore the fact that the rates are step functions and are not really continuous” (p. 376).

Working with discrete situations in systems of equations appears relevant for two reasons. First, it is possible that discrete functions, even if they have an intersection point when translated as continuous, do not intersect when plotted discretely (see **fig. 16**). A similar situation is illustrated in **figure 17**. Second, the study of discrete situations can lead to unexpected results. Contino (1995) illustrates this with two step functions (see **fig. 18**) with different steps that cross each other twice. (See Contino’s article for more details on the problem and its context and solution.) We are not arguing that situations identical to this one must be presented to students. Rather, students’ work on discrete situations can reveal intriguing instances requiring interpretive and critical skills to make sense of the situations. These strengthened skills, in turn, can enhance the study of systems of equations.

**CONCLUSION**

The concept map is intended to provide a more comprehensive perspective for studying systems of equations and invite reflection on how the teaching and study of systems of equations can be enhanced through this perspective. Presenting the meaning and relevance of using systems of equations and using multiple representations—multiple ways of solving and interpreting solutions of systems of equations—offers, in our opinion, some illustrations of this orientation.

Our experience as teachers and mathematics education researchers and our analyses of materials suggest that systems of equations is a topic most often approached through procedures, despite its
conceptual richness as a modeling device and as both an algebraic and graphical form. Systems of equations have important particularities, and their study should not be reduced solely to a series of mechanical steps or restricted to systems that intersect at one coordinate point. We believe that these perspectives, if taught in the classroom, can offer students rich mathematical experiences in developing robust understandings of systems of equations.

REFERENCES

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