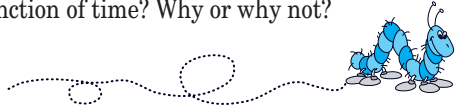


What Do Students Really Know about Functions?

How would your students respond to the following item?

A caterpillar is crawling around on a piece of graph paper, as shown below. If we wished to determine the creature's location on the paper with respect to time, would this location be a function of time? Why or why not?



Readers might be surprised to learn that 60 percent of the precalculus students who answered this item applied the vertical-line test directly to the path of the caterpillar to determine whether the caterpillar's location was a function of time, that is, that for every point in time, the caterpillar was at exactly one location. What does this response tell us about students' understanding of functions, the vertical-line test, and interpreting graphs? Read further to find out.

The concept of function plays an important role throughout the mathematics curriculum. The typical mathematics definition of *function* from x to y is a correspondence that associates with each element of x a unique element of y .

The concept of function is central to students' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs. Not surprisingly *Principles and Standards for School Mathematics* (NCTM 2000, p. 296) advocates instructional programs from prekindergarten through grade 12 that "enable all students to understand patterns, relations, and functions." Although the function concept is a central one in mathematics, many research studies of high school and college students have shown that it is also one of the most difficult for students to understand (Tall 1996; Sierpinski 1992; Markovits, Eylon, and Bruckheimer 1988; Dreyfus and Eisenberg 1982).

ALIGNMENT OF CONCEPT IMAGE WITH MATHEMATICAL DEFINITION

Vinner (1992) uses the term *concept image* to describe how students—as well as adults, for that

matter—think about concepts. A person's concept image consists of all the mental pictures that he or she associates with a given concept. A student's concept image can differ greatly from a mathematically acceptable definition; and students' concept images are often very narrow, or they may include erroneous assumptions.

A student's concept image of function may, for example, be limited to the graph of a relation that passes the vertical-line test or to a machine that furnishes an output when an input is supplied. Vinner identifies several other common aspects of students' concept images of functions. One is that students believe that a function should be systematic. "An arbitrary correspondence is not considered a function" (Vinner 1992, p. 200). Other common aspects of students' concept images include the following:

- A function should be given by a single rule. For example, a function with a split domain is often considered as two or more functions, depending on how the domain is split.
- The graph of a function should be continuous. For example, students do not generally consider the graph of the greatest-integer function to be a representation of a function.
- A function should be one-to-one, that is, functions have the additional property that for each element in the range, exactly one element exists in the domain. For example, $f(x) = 12$ is often not considered a function, since it is not one-to-one (Markovits, Eylon, and Bruckheimer 1988).

Precalculus students applied the vertical-line test directly to the path of the caterpillar

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These overly narrow views impede students' ability to determine functionality except in very specific prototypical instances.

Students often believe that a function must include some algebraic formula. However, some images that students have about the concept of function—for example, that a function must have an explicit analytical representation—correspond with historical concepts of functions (Tall 1996). Students may erroneously consider $y = \pm\sqrt{x^2 - 3}$ a function, since it is an algebraic formula; whereas they might not consider the correspondence that Mary owes \$6, John owes \$3, and Sue owes \$2 to be a function, since no formula “fits it.” Williams (1998) revealed that 26 percent of university students who were enrolled in a third-quarter calculus course persisted with a similar equation-bound view of functions. The students' concept images then prevented them from identifying some relations that actually are functions, whereas the students accepted as functions some relations that are not.

This article explores one theme related to understanding functions—developing a concept image that is well aligned with the mathematical definition. This exploration includes sharing assessment items that teachers can use to make sense of their students' thinking surrounding this theme, describing students' responses to these items, and interpreting those responses. Most of the assessment items were selected from items used in previous research studies. Twenty-eight such items were given at the end of the school year to thirty-five precalculus students who attended a public high school known for its academic excellence. Five of the thirty-five students were then interviewed. These students were selected to represent students who scored in the high (two students), middle (two students), and low (two students) ranges on the paper-and-pencil assessment. Readers may want to look at the five items included in **figure 1** before reading further, so that they can reflect on the items and view the percent of students who answered each item correctly.

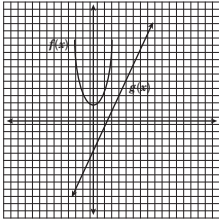
STUDENTS' RESPONSES TO ASSESSMENT ITEMS

Figure 1 presents items that are related to understanding students' concept images of functions. One item asked students to furnish their own definition of a *function*. This item allowed the author to assess students' understanding of function. Only four of the thirty-five precalculus students could give a definition that was similar to the mathematical definition of function, which recognizes that every element in the domain must be mapped to exactly one element in the range. One student wrote, “A function is a relation between two variables in which one variable is dependent upon the

Assessment Items to Determine Alignment of Students' Concept Image of Functions versus the Mathematical Definition

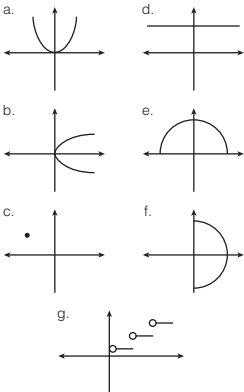
(The percent of students answering item correctly is listed in parentheses.)

- Given the following graphs of $f(x)$ and $g(x)$, graph $f(x) + g(x)$. (26%)



(Source: Lisa L. Clement, Alba Thompson, and Patrick Thompson)

- Circle the graphs that indicate that y is a function of x on the Cartesian coordinate system. Assume that the y -axis is the vertical axis. (40%)



(Source: “Horizontal and Vertical Growth of the Students' Conception of Function,” by Keith Schwindendorf, Julie Hawks, and Jennifer Beineke. In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 133–52. Washington, D.C.: Mathematical Association of America, 1992.)

- Which of the following indicate that y is a function of x ? Circle those that are functions. (51.4%)

a) $y = x^2 - 4$

d) $x^2 + y^2 = 25$

b) $y = x/2$

e) $y = e^x$

c) $xy = 8$

f) $y = \begin{cases} 1 & \text{if } x \in \text{rationals} \\ -1 & \text{otherwise} \end{cases}$

 g) If we let x = club member's name and y = amount owed, is y a function of x ?

Club Members' Dues Status			
Name	Owed	Name	Owed
Sue	\$17	Iris	6
John	6	Eve	12
Sam	27	Henry	14
Ellen	0	Louis	6

(Source for items 3(a) – 3(f) is “Teachers' Mathematical Knowledge of the Concept of Function,” by Alexander Norman, in *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 215–32 [Washington D.C.: Mathematical Association of America, 1992]. Source for item 3(g) is “The Nature of the Process Conception of Function,” by Ed Dubinsky and Guershon Harel, on pages 85–106 of the same work.)

- Define in your own words, the mathematical concept of function. (11.4%)

(Source: Lisa L. Clement, Alba Thompson, and Patrick Thompson)
- A caterpillar is crawling around on a piece of graph paper, as shown below. If we wished to determine the creature's location on the paper with respect to time, would this location be a function of time? Why or why not? Can time be described as a function of its location? Explain. (Two of the five students who were interviewed answered correctly.)

(Source: Lisa L. Clement, Alba Thompson, and Patrick Thompson)

Fig. 1

value of the other variable. For each independent variable, there must be only one value for the dependent variable.” More typical, however, was either an image of a machine—that is, when numbers are entered, numbers are produced, or an image of a graph that passes the vertical-line test. One student wrote, “A group of points which, when graphed, can have a vertical line drawn through them and touch the graph exactly once.” The responses of twenty of the thirty-five students, or 57 percent, were similar to one of the latter two responses.

Although these notions may initially help students think about characteristics of some functions, students who continue to hold only these images of functions may either become overly restrictive or not restrictive enough in deciding whether a particular expression represents a function. For example, students who think only of the vertical-line test when determining functionality may believe that expressions that they cannot view in graphical form are not functions. These students seem to make these decisions without ever considering a formal definition of *function*.

When students had to determine whether given relations were functions, they succeeded with items that are considered prototypes of functions and nonfunctions, as in items 2 and 3. For instance, in questions that asked whether y was a function of x , the majority of students correctly circled the graph of $y = x^2$ and correctly did not circle the graph of $x = y^2$. These graphs are often among the most common examples and nonexamples shown to students.

Students ran into more difficulty when the function given was not a prototype. For example, most students did not consider a graph of a noncontinuous (greatest-integer) function and a table showing club members’ names and the amount of money that each owed to be functions, even though they are both functions. In this situation, students were not using the actual mathematical definition of a *function* to determine functionality but were using their own concept image. The first set of students may have a concept image that includes the condition of continuity, whereas the second group of students may have a concept image that includes the condition of analytic representation.

During the interviews, students explained their responses to item 2, in which they had to determine whether certain graphs of relations were functions with respect to x . All five of the students who were interviewed said that they used the vertical-line test to determine whether each item was a function, and three of them stated something similar to the student who said, “If you have one certain value for x , it has to have one and only one y -value.” Pointing to (b), the student indicated, “This one

doesn’t work because this one x has two y -values.”

Students tended to recognize functions most often when the functions were familiar to them. That is, prototypes of functions were a greater factor in determining functionality than even the vertical-line test. One student used the vertical-line test but modified its use on the basis of her experience:

“Well, item (a) is a function because it is a parabola and it is always given as an example of a function.” Regarding the singleton point, she said, “I don’t think a point can be a function.” Finally, given a horizontal line, she stated, “I guess by my definition it would be, but I don’t think it is.” That student’s concept image seemed to involve a vertical-line test tempered by examples of functions that she had encountered in the past. Another student indicated that his concept image of function required that it pass both the vertical-line test and a condition of continuity. He stated that the greatest-integer function was not a function, since the pieces were not connected. The author asked, “So to be a function, does the graph have to be continuous?” He replied, “I think so; every one I have ever seen has been.”

As mentioned at the beginning of the article, three students who were interviewed about the caterpillar question in item 5 looked at the path of the caterpillar as a graph and determined that it was not a function on the basis of the vertical-line test. The other two students, however, drew location-versus-time graphs and determined that the caterpillar’s location was indeed a function with respect to time. A student explained, “As time progresses, the reason that it is a function is that you can’t go backwards in time, so even though the caterpillar circles around, that’s at a different time. . . .”

CONCLUSIONS

The concept of function is central to understanding mathematics, yet students’ understanding of functions appears either to be too narrowly focused or to include erroneous assumptions. These results suggest at least two implications for teaching and assessment.

First, although time is devoted to teaching functions (high school courses often even have the word *function* in the title), perhaps this time should be spent discussing functions in different ways. Some of the students’ overly narrow views of, or erroneous assumptions about, functions may be caused at least in part by what students see in textbooks as the prototypes of functions. Teachers also may often furnish students with examples of functions that fit the types of functions most often seen in textbooks. As a teacher, I certainly gave prototypical examples to my students as a way, I thought, to help them better understand the concept of functions. I was thus surprised when I found that the assessment items that I gave to students indicated

A graph of a greatest-integer function was not considered a function by most students

**The
interviews
shed much
more light on
how students
were
thinking
than did the
paper-and-
pencil
assessment
alone**

a large disparity between a mathematical definition of *function* and students' concept images of functions. When providing examples to my students, I never made explicit to them—or better yet, never had them make explicit for themselves—the aspects of the definition of *function* that they were attending to when they made determinations of functionality. We never discussed deeply the definition of function, the different ways to represent functions, and the connections between the two.

Second, assessment instruments—and ways of assessing students' thinking—should be changed to better determine what students understand about the underlying concept of function. The items in **figure 1** furnish some ways to assess students' concept image of functions, and the students' responses in this article indicate a range of possible student responses to the items. The items may also furnish an occasion for teachers to have mathematical conversations about functions with students. For example, the interviews with students shed much more light on how students were thinking than the paper-and-pencil assessment alone. Teachers can use the items to assess students before or after they have completed a unit, or they can use a variety of the items as questions for generating class discussion. The items are appropriate to use in courses ranging from algebra through calculus.

Although I have discussed in this article only one major theme related to understanding the concept of function—developing a concept image that is well aligned with the mathematical definition—teachers can begin to help their students understand func-

tions by assessing in multiple ways their current understandings and building from there. Best of luck in your journey to investigate your students' thinking about functions.

REFERENCES

- Dreyfus, Tommy, and Theodore Eisenberg. "Intuitive Functional Concepts: A Baseline Study on Intuitions." *Journal for Research in Mathematics Education* 13 (November 1982): 360–80.
- Dubinsky, Ed, and Guershon Harel. "The Nature of the Process Conception of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 85–106. Washington, D.C.: Mathematical Association of America, 1992.
- Markovits, Zvia, Bat Sheva Eylon, and Maxim Bruckheimer. "Difficulties Students Have with the Function Concept." In *The Ideas of Algebra, K–12*, 1988 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Arthur Coxford, pp. 43–60. Reston, Va.: NCTM, 1988.
- National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.
- Norman, Alexander. "Teachers' Mathematical Knowledge of the Concept of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 215–32. Washington, D.C.: Mathematical Association of America, 1992.
- Schwindgendorf, Keith, Julie Hawks, and Jennifer Beineke. "Horizontal and Vertical Growth of the Students' Conception of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 133–52. Washington, D.C.: Mathematical Association of America, 1992.
- Sierpinska, Anna. "On Understanding the Notion of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 25–58. Washington, D.C.: Mathematical Association of America, 1992.
- Tall, David. "Functions and Calculus." In *International Handbook of Mathematics Education, Part 1*, vol. 4, edited by Alan J. Bishop, Ken Clements, Christine Keitel, Jeremy Kilpatrick, and Colette Laborde, pp. 289–325. Dordrecht, Netherlands: Kluwer Academic Publishers, 1996.
- Vinner, Shlomo. "The Function Concept as a Prototype for Problems in Mathematics Learning." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 195–214. Washington, D.C.: Mathematical Association of America, 1992.
- Williams, Carol G. "Using Concept Maps to Assess Conceptual Knowledge of Functions." *Journal for Research in Mathematics Education* 29 (July 1998): 414–41.

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